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Christoffel Functions and Fourier Series for Multivariate Orthogonal Polynomials

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The Christoffel functions $\Lambda_n(d\mu)$ associated with a general nonnegative measure μ on $\mathbb{R}^d$ are studied. The asymptotics of $\Lambda_n(d\mu)$ are derived for μ supported on $[-1, 1]^d$. The estimates of $\Lambda_n(d\mu)$ are used to study the summability of the multivariate orthogonal polynomials associated with μ. The pointwise convergence of the partial sums of the orthogonal expansion, and their $(C, 1)$ and de la Vallée Poussin means are considered.

1. INTRODUCTION

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}_0^d$ we write $|\mathbf{k}| = k_1 + \cdots + k_d$. For $n \in \mathbb{N}_0$ we let Π_n^d be the set of polynomials of total degree n in d variables, and let Π^d be the set of all polynomials in d variables. For a nonnegative measure μ on $\mathbb{R}^d$ with finite moments, we let $\{P_{\mathbf{k}}^n\}_{|\mathbf{k}|=n}{}_{n=0}^{\infty}$, where $\mathbf{k} \in \mathbb{N}_0^d$ and $P_{\mathbf{k}}^n \in \Pi_n^d$, be a sequence of orthogonal polynomials associated with μ in d variables. For a function $f \in L_{d\mu}^2$, the nth partial sum of the orthogonal expansion of f is defined by

$$S_n(d\mu, f, \mathbf{x}) = \sum_{m=0}^{n-1} \sum_{|\mathbf{k}|=m} a_{\mathbf{k}}^m P_{\mathbf{k}}^m(\mathbf{x}), \qquad a_{\mathbf{k}}^m = \int_{\mathbb{R}^d} f(\mathbf{y})\, P_{\mathbf{k}}^m(\mathbf{y})\, d\mu.$$

The reproducing kernel of these multivariate orthogonal polynomials, denoted by $\mathbf{K}_n(d\mu, \cdot, \cdot)$, is

$$\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{y}) = \sum_{m=0}^{n-1} \sum_{|\mathbf{k}|=m} P_{\mathbf{k}}^m(\mathbf{x})\, P_{\mathbf{k}}^m(\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

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It serves as an integral kernel of $S_n(d\mu, f)$. The Christoffel function, denoted by $\Lambda_n(d\mu, \cdot)$, is defined by

$$\Lambda_n(d\mu, \mathbf{x}) = [\mathbf{K}_n(\mathbf{x}, \mathbf{x})]^{-1}.$$

The purpose of this paper is to study the asymptotic behavior of this function and the pointwise convergence of $S_n(d\mu, f)$ and its first Cesáro mean. While for the asymptotics we will restrict μ to a class supported on $[-1, 1]^d$, in general the measure is restrained by some conditions, but kept general otherwise.

For $d = 1$, the Christoffel function, usually denoted by $\lambda_n(d\alpha)$, plays a significant role in the study of orthogonal polynomials of one variable. One important property of the Christoffel functions states that

$$\lim_{n \to \infty} n\lambda_n(d\alpha, x) = \pi\alpha'(x)(1 - x^2)^{1/2} \qquad \text{for a.e.} \quad x \in [-1, 1] \tag{1.1}$$

for measures α belonging to Szegő's class, i.e., $\log \alpha'(\cos t) \in L^1[0, 2\pi]$ ([9, Theorem 5]). For the historical accounts concerning (1.1) and other properties of $\lambda_n(d\alpha)$, we refer to [2] and an extensive survey [11]. The summability of orthogonal polynomials with respect to a general measure α on $\mathbb{R}$ has been studied quite extensively. The order of $\lambda_n(d\alpha, x)$ as n goes to infinity is an important quantity in such a study, as demonstrated in the work of Freud (cf. [2, 11]). For example, suppose $[n\lambda_n(d\alpha, x)]^{-1}$ is bounded on a set Δ, then for a function f in $L^2_{d\alpha}$ the first Cesáro means of the orthogonal expansion of f converge almost everywhere to f on Δ. Moreover, the de la Vallée Poussin sums of the partial sums $S_k(d\alpha, f)$ converge to $f \in C$ with the same speed as the best polynomial approximation from Π_n (cf. [2, Chapt. 4] and [11, Sec. 4.4]).

Compared to the univariate results, the theory of multivariate orthogonal polynomials received comparatively little attention in the past. In fact, although many systems of orthogonal polynomials in several variables were known and studied for years, there are only a handful papers in the literature devoted to the study of the *general* theory. We refer to [1, Chapter XII] for the results before 1953, [4, 10, 13] for the development up to 80's. One of the major difficults lies in the fact that for each n there are $\binom{n+d-1}{n}$ linearly independent orthogonal polynomials of degree n in d variables which causes problems in ordering and in notation. Recently a vector notation has been introduced in order to overcome this difficulty (cf. [5, 6, 14–19]). Using this vector notation, many properties and theorems of orthogonal polynomials of one variable have been extended to multivariable, including the three-term relation, Favard's theorem, Christoffel–Darboux formula, Jacobi matrices, and many others. Important applications of this general theory are also found in the area of Gaussian cubature formulae. We refer to [14–19] and the references there. However, the present paper seems to be the first one dealing with the summability of multivariate orthogonal polynomials in general. We shall start with estimates of $\Lambda_n(d\mu, \mathbf{x})$ in Section 3. For μ that has compact support and subject to certain restraints, we have the order of $\Lambda(d\mu)$ as n^d. Naturally, a much stronger and highly desirable result would be the analogue of (1.1) for multivariate orthogonal polynomials. The limit relation (1.1) is usually proved by the use of Szegő's theory, which is established for complex orthogonal polynomials on the unit circle. The application of Szegő's theory on orthogonal polynomials of a real variable is carried out through the so-called *-transform, which plays a central role in the theory. While part of Szegő's theory can still be extended to the setting in several complex variables, nonetheless, our attempt to extend the *-transform in several variables has failed. The failure makes us believe that the extension of Szegő's theory to several variables is difficult, if possible at all. As a consequence, the usual method of proving (1.1) is not available at present time. Fortunately, a method in Freud [2] for orthogonal polynomials of one variable uses only real variables, which we shall extend to the multivariate setting. The measures are subject to certain additional restraints, but are still quite general. For one class of measures supported on $[-1, 1]^2$, we are able to establish in Section 4

$$\begin{aligned}\frac{\pi^2}{2^2} W(\mathbf{x}) \sqrt{1-x_1^2} \sqrt{1-x_2^2} &\leqslant \liminf_{n \to \infty} N_n \Lambda_n(d\mu, \mathbf{x}) \\ &\leqslant \limsup_{n \to \infty} N_n \Lambda_n(d\mu, \mathbf{x}) \\ &\leqslant 2^2 \pi^2 W(\mathbf{x}) \sqrt{1-x_1^2} \sqrt{1-x_2^2}. \qquad (1.2)\end{aligned}$$

for almost every $\mathbf{x}$ in $[-1, 1]^2$, where $N_n = \dim \Pi_{n-1}^2$. Although this result may still hold for $d > 2$, a somewhat unexpected behavior of the reproducing kernel of the product Chebyshev weight function has caused difficulties that we have not been able to overcome. Because of the dependence of the order of $\Lambda_n(d\mu)$ on the dimension, our results on the summability of orthogonal polynomials of d variables require stronger conditions on the class of functions being approximated for larger d. For example, for $d = 2$, the de la Vallée Poussin sums converge for functions in Lip^β, $\beta > 1/2$, instead of functions that are merely continuous as in the case of $d = 1$. These results and others on the summability of multivariate orthogonal polynomials are discussed in Section 5. Our proofs in this last part follow a method developed for orthogonal polynomials of one variable (cf. [2]); its extension to several variables is made possible by using the formulas developed in our recent papers, mainly [14, 15]. In the following section we fix the notation and give the necessary preliminaries.

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2. Preliminaries

Let $\mathcal{M}(\mathbb{R}^d)$ be the set of nonnegative measures μ on $\mathbb{R}^d$ whose moments are finite, i.e.

$$\int_{\mathbb{R}^d} |\mathbf{x}^{\mathbf{k}}| \, d\mu(\mathbf{x}) < \infty, \qquad \mathbf{k} \in \mathbb{N}_0^d,$$

where we have used the standard notation $\mathbf{x} = (x_1, ..., x_d)$ and $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_d^{k_d}$. Throughout this paper we denote by α the measure on $\mathbb{R}^1$ and by μ the measure on $\mathbb{R}^d$. If α is absolutely continuous, we write $d\alpha = \alpha'(x)\, dx$, and if μ is absolutely continuous with respect to the Lebesgue measure, we write $d\mu = W(\mathbf{x})\, d\mathbf{x}$, where W is the Radon–Nikodym derivative of μ. For convenience we always assume the measures are normalized by $\int d\mu = 1$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, let $P_{\mathbf{k}}^n(d\mu) \in \Pi_n^d$ be the orthogonal polynomials with respect to μ. For each n we arrange $P_{\mathbf{k}}^n$ according to the lexicographical order of the set $\{\mathbf{k} \in \mathbb{N}_0^d \colon |\mathbf{k}| = n\}$. Let $r_n = r_n^d = \dim \Pi_n^d - \dim \Pi_{n-1}^d$. Using vector notation

$$\mathbb{P}_n(d\mu, \mathbf{x}) = [P_{\mathbf{k}_1}^n(d\mu, \mathbf{x}), P_{\mathbf{k}_2}^n(d\mu, \mathbf{x}), ..., P_{\mathbf{k}_{r_n}}^n(d\mu, \mathbf{x})]^T$$

where the superscript T stands for the transpose and the elements are arranged according to the lexicographical order, we can express the orthonomality property of $\{P_{\mathbf{k}}^n(d\mu)\}$ by

$$\int \mathbb{P}_n(d\mu)\, \mathbb{P}_m^T(d\mu)\, d\mu = \delta_{m,n} I,$$

where I is the identity matrix of size $r_n \times r_n$ when $m = n$ and zero otherwise. For convenience, we call $\{\mathbb{P}_n(d\mu)\}_{n=0}^{\infty}$ the sequence of orthogonal polynomials. Using this vector notation, the multivariate orthogonal polynomials satisfy a three-term relation

$$x_i \mathbb{P}_n(d\mu) = A_{n,i} \mathbb{P}_{n+1}(d\mu) + B_{n,i} \mathbb{P}_n(d\mu) + A_{n-1,i}^T \mathbb{P}_{n-1}(d\mu), \qquad 1 \leqslant i \leqslant d, \tag{2.1}$$

where $\mathbb{P}_{-1}(d\mu) = 0$, $\mathbb{P}_0(d\mu) = 1$, and $A_{n,i}$ and $B_{n,i}$ are proper matrices of dimension $r_n \times r_{n+1}$ and $r_n \times r_n$, respectively, $A_{-1,i}$ is taken to be zero. Together with a rank condition on $A_{n,i}$ this relation characterizes the orthogonality of $\mathbb{P}_n$ (Favard's theorem [6, 14, 15]).

For a vector $\mathbf{a} \in \mathbb{R}^d$, we denote by $|\mathbf{a}|_2$ the l^2 norm, $|\mathbf{a}|_2 = \sqrt{\mathbf{a}^T \mathbf{a}}$. For a matrix A, we denote by $|A|_2$ the matrix norm induced by $|\mathbf{a}|_2$. In [15], using the spectral theorem of a commuting family of self-adjoint operators, we proved that $\sup_{n\geqslant 0}|A_{n,i}|_2<\infty$ and $\sup_{n\geqslant 0}|B_{n,i}|<\infty$, $1\leqslant i\leqslant d$, if and only if μ has compact support. One direction of this equivalence relation can also be proved as follows. From the three-term relation (2.1) and the orthogonality of $\mathbb{P}_n$, it follows that $A_{n,i}=\int x_i\mathbb{P}_n(d\mu,\mathbf{x})\,\mathbb{P}_{n+1}^T(d\mu,\mathbf{x})\,d\mu$. Therefore, if μ has compact support and $\operatorname{supp}\mu\subset[-M,M]^d$, then by the Cauchy-Schwarz inequality we have

$$\begin{aligned}|A_{n,i}|_2&\leqslant\left(\int|x_i|^2\,|\mathbb{P}_n(d\mu,\mathbf{x})|_2^2\,d\mu\right)^{1/2}\left(\int|\mathbb{P}_{n+1}(d\mu,\mathbf{x})|_2^2\,d\mu\right)^{1/2}\\&\leqslant M\left(\int|\mathbb{P}_n^T(d\mu,\mathbf{x})|_2^2\,d\mu\right)^{1/2}=M.\end{aligned}\tag{2.2}$$

A similar estimate can be given for $B_{n,i}=\int x_i\mathbb{P}_n(d\mu,\mathbf{x})\,\mathbb{P}_n^T(d\mu,\mathbf{x})\,d\mu$. Using the vector notation, the reproducing kernel of multivariate orthogonal polynomials takes the form

$$\mathbf{K}_n(d\mu,\mathbf{x},\mathbf{y})=\sum_{k=0}^{n-1}\mathbb{P}_k^T(d\mu,\mathbf{x})\,\mathbb{P}_k(d\mu,\mathbf{y})$$

and it has the property that gives its name

$$\int_{\mathbb{R}^d}P(\mathbf{y})\,\mathbf{K}_n(d\mu,\mathbf{y},\mathbf{x})\,d\mu(\mathbf{y})=P(\mathbf{x}),\qquad P\in\Pi_{n-1}^d.$$

From the three-term relation follows the analogue of the Christoffel–Darboux formula ([14])

$$\mathbf{K}_n(d\mu,\mathbf{x},\mathbf{y})=\frac{[A_{n-1,i}\mathbb{P}_n(d\mu,\mathbf{x})]^T\,\mathbb{P}_{n-1}(d\mu,\mathbf{y})-\mathbb{P}_{n-1}^T(d\mu,\mathbf{x})[A_{n-1,i}\mathbb{P}_n(d\mu,\mathbf{y})]}{x_i-y_i}\tag{2.3}$$

for $1\leqslant i\leqslant d$. Let $L^2_{d\mu}$ denote the space of μ-measurable functions for which

$$\|f\|_{d\mu,2}:=\left\{\int_{\mathbb{R}^d}|f(\mathbf{x})|^2\,d\mu(\mathbf{x})\right\}^{1/2}<\infty.$$

For $f\in L^2_{d\mu}$, we consider its multivariate orthogonal polynomial expansion, which, in vector notation, is

$$f\sim\sum_{k=0}^{\infty}\mathbf{a}_k^T(d\mu,f)\,\mathbb{P}_k(d\mu),\qquad \mathbf{a}_k(d\mu,f)=\int_{\mathbb{R}^d}f(\mathbf{y})\,\mathbb{P}_k(d\mu,\mathbf{y})\,d\mu.$$

The vectors $\mathbf{a}_k(d\mu, f)$ are called the Fourier coefficients of f. The nth partial sum of this expansion, denoted by $S_n(d\mu, f)$, is defined by

$$S_n(d\mu, f, \mathbf{x}) = \sum_{k=0}^{n-1} \mathbf{a}_k^T(d\mu, f)\, \mathbb{P}_k(d\mu, \mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{y})\, \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{y})\, d\mu(\mathbf{y}). \quad (2.4)$$

If Π^d is dense in $L^2_{d\mu}$, then we have the Parseval identity

$$\sum_{k=0}^{\infty} |\mathbf{a}_k(d\mu, f)|_2^2 = \int_{\mathbb{R}^d} |f(\mathbf{x})|^2\, d\mu(\mathbf{x}). \quad (2.5)$$

The Christoffel function is defined by

$$\Lambda_n(d\mu, \mathbf{x}) = [\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x})]^{-1} = \left[\sum_{k=0}^{n-1} |\mathbb{P}_k(d\mu, \mathbf{x})|_2^2\right]^{-1}. \quad (2.6)$$

It satisfies the following fundamental property ([14])

$$\Lambda_n(\mathbf{x}) = \min_{\substack{P \in \Pi_{n-1}^d \\ P(\mathbf{x}) = 1}} \int_{\mathbb{R}^d} |P(\mathbf{y})|^2\, d\mu(\mathbf{y}). \quad (2.7)$$

For other general properties of multivariate orthogonal polynomials we refer to [5, 6, 10, 13, 14–19].

Throughout this paper, for $d = 1$ and the measure $\alpha \in \mathcal{M}(\mathbb{R}^1)$, we write $K_n(d\alpha, x, y)$ and $\lambda_n(d\alpha, x)$ for the reproducing kernel and the Christoffel function of orthogonal polynomials of one variable. We use the letter Δ to denote a set in either $\mathbb{R}^d$ or $\mathbb{R}$ whose exact meaning will be given locally. The constants in this paper follow the following rules. We use the letters c and $c_1, c_2, \ldots$ for constant that depend only on the dimension d and other fixed parameters, their values may vary from place to place. The letters M and $M_1, M_2, \ldots$ denote constants that retain their values.

3. The Estimate of the Christoffel Function

We consider a simple situation here; its corollary (Corollary 3.2) nevertheless illustrates the behavior of $\Lambda_n(\mathbf{x})$.

Theorem 3.1. *Let $\alpha \in \mathcal{M}(\mathbb{R}^1)$ be absolutely continuous, and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be absolutely continuous with respect to Lebesgue measure, $d\mu = W d\mathbf{x}$. If*

$$W(\mathbf{x}) \geq c_1 \prod_{i=1}^{d} \alpha'(x_i), \quad (3.1)$$

then

$$\Lambda_n(d\mu, \mathbf{x}) \geqslant c_1 \prod_{i=1}^{d} \lambda_n(d\alpha, x_i). \tag{3.2}$$

On the other hand, if

$$W(\mathbf{x}) \leqslant c_2 \prod_{i=1}^{d} \alpha'(x_i), \tag{3.3}$$

then

$$\Lambda_n(d\mu, \mathbf{x}) \leqslant c_2 \prod_{i=1}^{d} \lambda_{[(n-1)/d]}(d\alpha, x_i) \tag{3.4}$$

Proof. Let $d\mu_0 = W_0 d\mathbf{x}$ with $W_0(\mathbf{x}) = \prod_{i=1}^{d} \alpha'(x_i)$. Then the orthonormal polynomial $P_{\mathbf{k}}^n(d\mu_0)$, $|\mathbf{k}| = n$, is equal to $p_{k_1}(d\alpha) \cdots p_{k_d}(d\alpha)$. Therefore, by (2.7) and (3.1)

$$\begin{aligned}
\Lambda_n(d\mu, \mathbf{x}) &\geqslant \Lambda_n(c_1 d\mu_0, \mathbf{x}) = c_1 \Lambda_n(\mu_0, \mathbf{x}) \\
&= c_1 \left[\sum_{m=1}^{n-1} \sum_{|\mathbf{k}|=m} [P_{\mathbf{k}}^n(d\mu_0, \mathbf{x})]^2 \right]^{-1} \\
&\geqslant c_1 \left[\sum_{k_1=1}^{n-1} \cdots \sum_{k_d=1}^{n-1} p_{k_1}^2(d\alpha, x_1) \cdots p_{k_d}^2(d\alpha, x_d) \right]^{-1} \\
&= c_1 \prod_{i=1}^{d} [K_n(d\alpha, x_i)]^{-1} = c_1 \prod_{i=1}^{d} \lambda_n(d\alpha, x_i).
\end{aligned}$$

This proves the inequality (3.2). On the other hand, suppose (3.3) holds. Let p_x be a polynomial of one variable of degree at most $n-1$, such that $p_x(x) = 1$ and $\lambda_n(d\alpha, x) = \int p_x^2(t)\, d\alpha(t)$. Let $P = p_{x_1} \cdots p_{x_d} \in \Pi_{d(n-1)}^d$. Then $P(\mathbf{x}) = 1$ and we have by (2.7)

$$\begin{aligned}
\Lambda_{d(n-1)+1}(d\mu, \mathbf{x}) &\leqslant \Lambda_{d(n-1)+1}(c_2 d\mu_0, \mathbf{x}) = c_2 \Lambda_{d(n-1)+1}(d\mu_0, \mathbf{x}) \\
&\leqslant c_2 \int_{\mathbb{R}^d} P^2(\mathbf{x})\, d\mu = c_2 \prod_{i=1}^{d} \int_{\mathbb{R}^1} p_{x_i}^2(t)\, d\alpha \\
&= c_2 \prod_{i=1}^{d} \lambda_n(d\alpha, x_i).
\end{aligned}$$

Since $\Lambda_n(\mu)$ is nonincreasing, changing $d(n-1)$ to n proves the inequality (3.4). ∎

COROLLARY 3.2. *Let $\alpha \in \mathcal{M}(\mathbb{R}^1)$ be absolutely continuous and* $\operatorname{supp} \alpha \subseteq [-1, 1]$. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure. If* (3.1) *holds and*

$$[n\lambda_n(\alpha, x_i)]^{-1} \leqslant c_3, \qquad 1 \leqslant i \leqslant d, \tag{3.5}$$

then

$$[n^d \Lambda_n(\mu, \mathbf{x})]^{-1} \leqslant c_5, \tag{3.6}$$

where $c_5 = c_1^{-1} c_3^d$. On the other hand, if (3.3) *holds and*

$$n\lambda_n(\alpha, x_i) \leqslant c_4, \qquad 1 \leqslant i \leqslant d \tag{3.7}$$

then

$$n^d \Lambda_n(\mu, \mathbf{x}) \leqslant c_6, \tag{3.8}$$

where $c_6 = c_2(dc_4)^d$. Moreover, if (3.1) *and* (3.5), *or* (3.3) *and* (3.7), *hold uniformly on a set Δ, then so does* (3.6) *or* (3.8).

This corollary follows from Theorem 3.1 easily. Conditions (3.5) and (3.7) have been established for many classes of measures (cf. [2, 9] and the references there). In particular, for measures in Szegő's class, they follow from (1.1) for almost every x in $[-1, 1]$. If, in addition, α is absolutely continuous and α' is continuous and positive on an interval $\Delta \subset [-1, 1]$, then (1.1) holds uniformly for $x \in \Delta$ (cf. [11, p. 18]). Using these results, we can easily write down several specail cases of Corollary 3.2. For example, for the product measure $d\mu = \alpha'(x_1) \cdots \alpha'(x_d)\, d\mathbf{x}$ on $[-1, 1]^d$, where α' satisfies the conditions for which (1.1) holds uniformly on Δ, we have that (3.6) and (3.8) hold uniformly on Δ^d. We shall not fomulate further consequences of Corollary 3.2 explicitly, instead we consider several classical weight functions to illustrate the results. Let $w^{(a,b)}$ be the Jacobi weight function, $w^{(a,b)}(x) = x^a(1-x)^b$, $a, b > -1$, on $(0, 1)$. It is known (cf. [11, p. 18]) that

$$c_1 \leqslant n\lambda_n(w^{(a,b)}, x) \leqslant c_2$$

for every $x \in (0, 1)$, and the inequalities hold uniformly for every closed interval $\Delta \subset (0, 1)$. Let $d\mu = W(\mathbf{x})\, d\mathbf{x}$.

EXAMPLE 1. $W(\mathbf{x}) = w^{(a_1,b_1)}(x_1) \cdots w^{(a_d,b_d)}(x_d)$ on the cube $\Omega = [0, 1]^d$.

EXAMPLE 2. $W(\mathbf{x}) = x_1^{a_1} \cdots x_d^{a_d}(1 - |\mathbf{x}|)^b$, where $|\mathbf{x}| = x_1 + \cdots + x_d$, on the simplex $\Omega = \{\mathbf{x} : x_1 \geqslant 0, ..., x_d \geqslant 0, \text{ and } 1 - |x| \geqslant 0\}$.

EXAMPLE 3. $W(\mathbf{x}) = (1 - |\mathbf{x}|_2^2)^b$, where $|\mathbf{x}|_2^2 = x_1^2 + \cdots + x_d^2$, on the ball $\Omega = \{\mathbf{x}: |x|_2 \leqslant 1\}$.

The orthogonal polynomials for these three cases are introduced in connection with hypergeometric series of several variables (cf. [1, Chapter 12]). They are eigenfunctions of certain second order partial differential equations, see [1] and [7] for the case $d=2$. Let $\Delta \subset \Omega$ be compact and the distance from Δ to the boundary of Ω be positive. Since the only zeros or singular points of W in these examples are on the boundary of Ω, for $\mathbf{x} \in \Delta$ we can use Theorem 3.1 and Corollary 3.2 by comparing W with the function that takes the value 1 for every point in $[-1, 1]^d$ and zero outside. This function is the product of copies of $w^{(0,0)}$, therefore, we have that in all these cases, (3.6) and (3.8) hold uniformly for $\mathbf{x} \in \Delta$.

4. Asymptotics of the Christoffel Functions

In this section, we prove (1.2) for a class of measures supported on $[-1, 1]^2$. Our method depnds on a thorough investigation of the reproducing kernel for the product Chebyshev weight function W_0. In Section 4.1, we estimate the Christoffel functions $\Lambda_n(W_0)$. A compact formula for the reproducing kernel $\mathbf{K}_n(W_0, \cdot, \cdot)$ is given in Section 4.2, and used to derive the estimates on $\mathbf{K}_n(W_0)$. The main result on the asymptotics of the Christoffel functions is proved in Section 4.3.

4.1. *The Product Chebyshev Measure*

We denote the classical Chebyshev measure by α_0, which is defined by

$$\alpha_0' = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, \qquad x \in [-1, 1],$$

and zero outside $[-1, 1]$. By the product Chebyshev measure on $[-1, 1]^d$ we mean the measure μ_0 defined by $d\mu_0 = W_0(\mathbf{x})\, d\mathbf{x}$ where

$$W_0(\mathbf{x}) = \alpha_0'(x_1) \cdots \alpha_d'(x_d) = \frac{1}{\pi^d} \prod_{i=1}^{d} \frac{1}{\sqrt{1-x_i^2}}, \qquad x \in [-1, 1].$$

In the definition of α_0 we already incorporate the factor π^{-1}, so that the integral of α_0' on $[-1, 1]$ is 1. The orthonormal polynomials with respect to α_0 are

$$T_0(x) = 1, \qquad T_k(x) = \sqrt{2} \cos k\theta, \qquad k \geqslant 1, \qquad x = \cos\theta. \quad (4.1.1)$$

We warn the reader that our definition of Chebyshev polynomials differs from the usual one by a factor $\sqrt{2}$, since our T_n are orthonormal with respect to α_0. It is easy to verify that the orthogonal polynomials with respect to μ_0 are given by

$$P_{\mathbf{k}}^n(d\mu_0) = T_{k_1}(x_1) \cdots T_{k_d}(x_d), \qquad |\mathbf{k}| = n, \qquad \mathbf{k} \in \mathbb{N}_0^d. \quad (4.1.2)$$

We need some properties of the reproducing kernel of α_0,

$$K_n(d\alpha_0, x, y) = \sum_{k=0}^{n-1} T_k(x)\, T_k(y),$$

which we summarize in the following.

LEMMA 4.1.1. *Let $x = \cos\theta$ and $y = \cos\phi$, $0 \leqslant \theta, \phi \leqslant \pi$. Then for $n \geqslant 1$*

$$|K_n(d\alpha_0, x, y)| \leqslant 4 \min\{n, |\theta - \phi|^{-1}\} \leqslant 4 \min\{n, |x - y|^{-1}\} \quad (4.1.3)$$

and

$$K_n(d\alpha_0, x, x) = n - \frac{1}{2} + \frac{\sin(2n-1)\theta}{2 \sin\theta} \geqslant \frac{n}{2}. \quad (4.1.4)$$

This lemma can be proved by using the explicit formula of $K_n(d\alpha_0)$ which is available through the summation of trigonometric functions. We refer to [2, p. 101, and p. 175]. The inequality (4.1.4) is stated in [2] for $n \geqslant 3$, but the case $n = 1$ and 2 can easily be verified. The estimate (4.1.3) will be used only in the next section.

From (4.1.4) one immediately has the following

$$\lim_{n \to \infty} n^{-1} K_n(d\alpha_0, x, x) = 1, \qquad x \in (-1, 1),$$

which is equivalent to (1.1) for α_0. For $d > 1$, however, the nice formula in (4.1.4) is not available anymore. Indeed, even for $d = 2$ we are able to find the explicit formula only after a tedious computation. We give this formula in the following to explain the difficulties, but leave out the deduction.

PROPOSITION 4.1.2. *Let $d = 2$, $\mathbf{x} = (x_1, x_2)$, and $x_1 = \cos\theta$, $x_2 = \cos\phi$. Then*

$$\begin{aligned}
\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x}) \\
&= \frac{1}{2}\left(n^2 - n + \frac{1}{2}\right) \\
&\quad + \frac{1}{4}\left(\frac{\sin(2n-1)\theta}{\sin\theta} + \frac{\sin(2n-1)\phi}{\sin\phi}\right) \\
&\quad + \frac{1}{2}\left(\frac{\sin^2(n-1)\theta}{\sin^2\theta} + \frac{\sin^2(n-1)\phi}{\sin^2\phi}\right) \\
&\quad + \frac{1}{4}\left(\frac{\sin n(\theta+\phi)}{\sin(\theta+\phi)} \frac{\sin n(\theta-\phi)}{\sin(\theta-\phi)} + \frac{\sin(n-1)(\theta+\phi)}{\sin(\theta+\phi)} \frac{\sin(n-1)(\theta-\phi)}{\sin(\theta-\phi)}\right).
\end{aligned}$$

Fortunately, to prove the limit relation analog to (1.1) for W_0, we do not need the explicit formula of $\mathbf{K}_n(d\mu_0)$, the following lemma is enough.

LEMMA 4.1.3. *For $n \geqslant 1$ and $\mathbf{x} \in \mathbb{R}^d$,*

$$\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x}) = \binom{n+d-1}{d} + R_n(\mathbf{x}), \tag{4.1.5}$$

where R_n satisfies the following properties

$$|R_n(\mathbf{x})| \leqslant cn^{d-1} \prod_{i=1}^{d} \left(1 + \frac{1}{\sqrt{1-x_i^2}}\right), \qquad \mathbf{x} \in (-1, 1)^d \tag{4.1.6}$$

and

$$\int_{[-1,1]^d} |R_n(\mathbf{x})| \, d\mu_0 \leqslant cn^{d-1} \log n. \tag{4.1.7}$$

Proof. We use induction on d. To indicate the dependence on the dimension, we write $\mathbf{K}_{n,d}(d\mu_0)$ and $R_{n,d} = R_n$. For $d=1$, we have by (4.1.4)

$$R_{n,1} = -\frac{1}{2} + \frac{\sin(2n-1)\,\theta}{2\sin\theta}.$$

Clearly, for every $x \in (-1, 1)$, we have

$$|R_{n,1}(x)| \leqslant \frac{1}{2} + \frac{1}{2\sqrt{1-x^2}}.$$

The inequality (4.1.7) for $d=1$ follows from the formula of $R_{n,1}$ by a change of variable and the use of a well-known inequaltity

$$\frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin(n-1/2)\,\theta}{\sin(\theta/2)}\right| d\theta < 3 + \frac{4}{\pi^2}\log n. \tag{4.1.8}$$

Assume now that the theorem has been proved for $\mathbf{K}_{n,d-1}(d\mu_0)$. For $\mathbf{x} \in \mathbb{R}^d$, we write $\mathbf{x} = (x_1, \mathbf{x}')$ where $\mathbf{x}' = (x_2, ..., x_d)$. Moreover, let $x_1 = \cos\theta$, $0 \leqslant \theta \leqslant \pi$. By (4.1.2) we find

$$\mathbf{K}_{n,d}(d\mu_0, \mathbf{x}, \mathbf{x}) = \sum_{j=0}^{n-1} T_j^2(x_1)\, \mathbf{K}_{n-j,d-1}(d\mu_0, \mathbf{x}', \mathbf{x}').$$

Therefore, by induction hypothesis, we can write

$$\begin{aligned}\mathbf{K}_{n,d}(d\mu_0, \mathbf{x}, \mathbf{x}) &= \sum_{j=0}^{n-1} T_j^2(x_1)\left[\binom{n-j+d-2}{d-1} + R_{n-j,d-1}(\mathbf{x}')\right] \\ &= \binom{n+d-2}{d-1} + 2\sum_{j=1}^{n-1} \cos^2 j\theta \binom{n-j+d-2}{d-1} \\ &\quad + \sum_{j=0}^{n-1} T_j^2(x_1)\, R_{n-j,d-1}(\mathbf{x}') \\ &= \sum_{j=0}^{n-1} \binom{n-j+d-2}{d-1} + \sum_{j=1}^{n-1} \cos 2j\theta \binom{n-j+d-2}{d-1} \\ &\quad + \sum_{j=0}^{n-1} T_j^2(x_1)\, R_{n-j,d-1}(\mathbf{x}').\end{aligned}$$

Using the identity

$$\sum_{j=0}^{n-1} \binom{n-j+d-2}{d-1} = \sum_{j=0}^{n-1} \binom{j+d-1}{d-1} = \binom{n+d-1}{d},$$

which is known and can easily be proved by induction on n, then we have

$$\mathbf{K}_{n,d}(d\mu_0, \mathbf{x}, \mathbf{x}) = \binom{n+d-1}{d} + R_{n,d}(\mathbf{x})$$

where

$$\begin{aligned}R_{n,d}(\mathbf{x}) &= \sum_{j=1}^{n-1} \cos 2j\theta \binom{n-j+d-2}{d-1} + \sum_{j=0}^{n-1} T_j^2(x_1)\, R_{n-j,d-1}(\mathbf{x}') \\ &:= I_1(x_1) + I_2(\mathbf{x})\end{aligned}$$

We need to prove that $R_{n,d}$ satisfies both (4.1.6) and (4.1.7). Let

$$B_d(x) = \prod_{i=1}^{d} \left(1 + \frac{1}{\sqrt{1-x_i^2}}\right).$$

By induction hypothesis and (4.1.4), we have

$$\begin{aligned}|I_2(\mathbf{x})| &\leqslant c B_{d-1}(\mathbf{x}')\, n^{d-2} \sum_{j=0}^{n-1} T_j^2(x_1) \\ &\leqslant c B_{d-1}(\mathbf{x}')\, n^{d-2}\left(n - \frac{1}{2} + \frac{1}{2\sqrt{1-x_1^2}}\right) \leqslant c B_d(\mathbf{x})\, n^{d-1}.\end{aligned}$$

and

$$\int_{[-1,1]^d} |I_2(\mathbf{x})|\, d\mu_0 \leqslant c \int_{-1}^{1} \sum_{j=0}^{n-1} T_j^2(x_1)\, d\alpha_0(x_1)$$

$$\times \int_{[-1,1]^{d-1}} |R_{n-j,d-1}(\mathbf{x}')|\, d\mu_0(\mathbf{x}')$$

$$\leqslant c \sum_{j=0}^{n-1} (n-j)^{d-2} \log(n-j) \leqslant cn^{d-1} \log n.$$

Therefore, we only need to show that $I_1(x_1)$ satisfies the same bounds. The estimates are based on the following observation. Let the kth Cesáro sums of a sequence $s_0, s_1, \ldots$ be defined by

$$S_k^0 = s_k, \qquad S_k^\beta = S_0^{\beta-1} + \cdots + S_k^{\beta-1}, \qquad \beta \in \mathbb{N}_0.$$

This notation of S_k^β is consistent with [20, p. 75]; it is used only in the rest of this poof. Then

$$\sum_{j=0}^{n-1} \cos j\theta \binom{n-j+d-2}{d-1} = S_{n-1}^d(\theta)$$

where $S_n^\beta(\theta)$ is the nth Cesáro sum of the βth order of the sequence 1, $\cos\theta$, $\cos 2\theta$, ... Indeed, using the formula of Cesáro sums as in [20, p. 77], this formula can be verified easily. We notice that our first Cesáro sum of this sequence is just

$$S_{n-1}(\theta) = S_{n-1}^1(\theta) = \sum_{k=0}^{n-1} \cos k\theta = \frac{1}{2} + \frac{\sin(n-1/2)\,\theta}{2\sin(\theta/2)}.$$

From the definition of I_1, we have

$$I_1(x_1) = \sum_{j=0}^{n-1} \cos 2j\theta \binom{n-j+d-2}{d-1} - \binom{n+d-2}{d-1}$$

$$= S_{n-1}^d(2\theta) - \binom{n+d-2}{d-1}.$$

Since (cf. [20, p. 77])

$$\binom{n+\beta}{\beta} = \frac{n^\beta}{\Gamma(\beta+1)}(1+O(n^{-1})), \tag{4.1.9}$$

where Γ is the Gamma function, we only need to estimate $I_3(x_1) := S_{n-1}^d(2\theta)$ to conclude the proof. By the definition of the Cesáro sums, we clearly have

$$|S_{n-1}^d(\theta)| \leqslant n^{d-1} \max_{0 \leqslant k \leqslant n-1} |S_k^1(\theta)| \leqslant cn^{d-1} \max_{0 \leqslant k \leqslant n-1} \left(1 + \frac{\sin(2k-1)\theta}{\sin\theta}\right).$$

In particular, we have

$$|I_3(x_1)| \leqslant cn^{d-1}\left(1 + \frac{1}{\sqrt{1-x_1^2}}\right) \leqslant cn^{d-1} B_d(\mathbf{x}),$$

and by (4.1.8),

$$\int_{[-1,1]^d} |I_3(x_1)|\, d\mu_0 \leqslant cn^{d-1}\left(1 + \max_{0 \leqslant k \leqslant n-1} \frac{1}{\pi}\int_0^\pi \left|\frac{\sin(2k-1)\theta}{\sin\theta}\right| d\theta\right)$$
$$\leqslant cn^{d-1} \log n.$$

Putting the estimates of I_1, I_2, and I_3 together completes the induction. ∎

This lemma contains more than what we need to prove the limit relation for $\Lambda_n(W_0)$; the inequality (4.1.7) is for later use in Section 4.3. Using (4.1.6) in this lemma we easily obtain the limit relation,

THEOREM 4.1.4. *For $d \geqslant 1$, and $\mathbf{x} \in (-1, 1)^d$,*

$$\lim_{n \to \infty} \binom{n+d-1}{d} \Lambda_n(d\mu_0, \mathbf{x}) = 1.$$

Proof. From (4.1.5) and (4.1.6) we have

$$\left|\frac{1}{\binom{n+d-1}{d}} \mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x}) - 1\right| \leqslant cn^{-d}|R_n(\mathbf{x})| \leqslant cn^{-1} \prod_{i=1}^d \left(1 + \frac{1}{\sqrt{1-x_i^2}}\right),$$

for every $\mathbf{x} \in (-1, 1)^d$. Therefore,

$$\lim_{n \to \infty} \frac{1}{\binom{n+d-1}{d}} \mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x}) = 1,$$

which implies the desired result. ∎

4.2. *A Compact Formula for the Kernel Function*

In this section we derive a compact formula for the reproducing kernel function $\mathbf{K}_n(d\mu_0)$. This formula will be used in the following section, and is of interest in itself. For $\mathbf{x} \in [-1, 1]^d$ we use the notation $x_i = \cos \theta_i$, and we denote by Θ the vector $\Theta = (\theta_1, ..., \theta_d)$, and $\Phi = (\phi_1, ..., \phi_d)$. Moreover, we use the notation $\cos \Theta$ to denote the vector $\cos \Theta = (\cos \theta_1, ..., \cos \theta_d)$. For $\mathbf{x} = \cos \Theta$ we define

$$L_n(\mathbf{x}) = \sum_{|\mathbf{k}| \leqslant n-1}' \cos k_1\theta_1 \cdots \cos k_d\theta_d, \tag{4.2.1}$$

where the notation $\sum'$ means that whenever a $k_i = 0$ the term containing $\cos k_i\theta_i$ is halved.

LEMMA 4.2.1. *Let* $\mathbf{x}, \mathbf{y} \in [-1, 1]^d$, *and* $\mathbf{x} = \cos \Theta$ *and* $\mathbf{y} = \cos \Phi$. *Let* ε *be a sign vector,* $\varepsilon \in \{-1, 1\}^d$, *and denote by* $\Phi + \varepsilon\Theta$ *the vector that has components* $\phi_i + \varepsilon_i\theta_i$. *Then*

$$\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{y}) = \sum_{\varepsilon \in \{-1, 1\}^d} L_n(\cos(\Phi + \varepsilon\Theta)) \tag{4.2.2}$$

where the summation is over all $\varepsilon \in \{-1, 1\}^d$.

Proof. From the definition of $\mathbf{K}_n(d\mu_0)$ we have

$$\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{y}) = 2^d \sum_{|\mathbf{k}| \leqslant n-1}' \cos k_1\theta_1 \cos k_1\phi_1 \cdots \cos k_d\theta_d \cos k_d\phi_d,$$

Using the trigonometric identity

$$\cos k\theta \cos k\phi = \tfrac{1}{2} [\cos k(\theta + \phi) + \cos k(\theta - \phi)],$$

the desired identity follows easily. ∎

Next we establish a compact formula for the function $L_n(\mathbf{x})$. For this purpose, we need the following trigonometric identities.

LEMMA 4.2.2. *Let* $0 \leqslant \theta, \phi \leqslant \pi$. *Then*

$$\sum_{k=0}^{n-1}{}' \cos k\theta \sin\left(n - k - \frac{1}{2}\right)\phi = \sin\frac{\phi}{2}\left[\frac{\cos\left(n-\frac{1}{2}\right)\phi\cos\frac{\phi}{2} - \cos\left(n-\frac{1}{2}\right)\theta\cos\frac{\theta}{2}}{\cos\phi - \cos\theta}\right], \tag{4.2.3}$$

and

$$\sum_{k=0}^{n-1}{}' \cos k\theta \cos\left(n - k - \frac{1}{2}\right)\phi = \cos\frac{\phi}{2}\left[\frac{\sin\left(n-\frac{1}{2}\right)\phi\sin\frac{\phi}{2} - \sin\left(n-\frac{1}{2}\right)\theta\sin\frac{\theta}{2}}{\cos\theta - \cos\phi}\right]. \tag{4.2.4}$$

Proof. These two identities are proved by using simple trigonometric identities, we only give a sketch of the first one. We write

$$\begin{aligned}\sum_{k=0}^{n-1}{}' \cos k\theta \sin\left(n-k-\frac{1}{2}\right)\phi &= \sin\left(n-\frac{1}{2}\right)\phi \sum_{k=0}^{n-1}{}' \cos k\theta \cos k\phi \\ &\quad -\cos\left(n-\frac{1}{2}\right)\phi \sum_{k=0}^{n-1}{}' \cos k\theta \sin k\phi,\end{aligned}$$

and use the sum of cosines and sum of sines to replace the products $\cos k\theta \cos k\phi$ and $\cos k\theta \sin k\phi$. Then we use the well-known identities

$$\sum_{k=0}^{n-1}{}' \cos k\theta = \frac{\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \sum_{k=1}^{n-1} \sin k\theta = \frac{\cos\frac{\theta}{2} - \cos\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

on the sums, which leads to an expression that can be simplified by the use of the elementary trigonometric identities to the desired formula. ∎

THEOREM 4.2.3. *For $d > 1$, $\Theta \in [0, \pi]^d$,*

$$L_n(\cos\Theta) = \frac{(-1)^{[(d-1)/2]}}{2^{d-1}} \sum_{i=1}^{d} \frac{\cos\frac{\theta_i}{2}(\sin\theta_i)^{d-2} F_n(\theta_i)}{\prod_{j=1, i\neq j}^{d}(\cos\theta_i - \cos\theta_j)}, \tag{4.2.5}$$

where $F_n(\theta) = \sin(n-1/2)\,\theta$ for odd d, and $F_n(\theta) = \cos(n-1/2)\,\theta$ for even d.

Proof. We use induction on d. To indicate the dependence on d we write $L_{n,d} = L_n$. For $\Theta \in [0, \pi]^{d+1}$, we write $\Theta = (\Theta', \theta_{d+1})$ with $\Theta' \in [0, \pi]^d$. Our induction process is based on the following relation

$$L_{n,d+1}(\cos\Theta) = \sum_{k=0}^{n-1}{}' \cos k\theta_{d+1} L_{n-k,d}(\cos\Theta') \tag{4.2.6}$$

and the well-known fact that

$$L_{n,1}(\cos\theta) = \frac{\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}.$$

From these, using the identities in Lemma 4.2.2, we easily deduce that

$$L_{n,2}(\cos\Theta)=\frac{1}{2}\,\frac{\cos\left(n-\frac{1}{2}\right)\theta_1\cos\frac{\theta_1}{2}-\cos\left(n-\frac{1}{2}\right)\theta_2\cos\frac{\theta_2}{2}}{\cos\theta_1-\cos\theta_2} \tag{4.2.7}$$

and

$$\begin{aligned}L_{n,3}(\cos\Theta)=-\frac{1}{4}\Bigg[&\frac{\sin\left(n-\frac{1}{2}\right)\theta_1\cos\frac{\theta_1}{2}\sin\theta_1}{(\cos\theta_2-\cos\theta_1)(\cos\theta_3-\cos\theta_1)}\\&+\frac{\sin\left(n-\frac{1}{2}\right)\theta_2\cos\frac{\theta_2}{2}\sin\theta_2}{(\cos\theta_1-\cos\theta_2)(\cos\theta_3-\cos\theta_2)}\\&+\frac{\sin\left(n-\frac{1}{2}\right)\theta_3\cos\frac{\theta_3}{2}\sin\theta_3}{(\cos\theta_1-\cos\theta_3)(\cos\theta_2-\cos\theta_3)}\Bigg].\end{aligned}$$

Assume now that the theorem has been proved for $L_{n,r}$ with $r\leqslant d$. We prove it for $L_{n,d+1}$. First, suppose that d is odd. Using the relation (4.2.6), the induction hypothesis, and (4.2.3), we have

$$\begin{aligned}&L_{n,d+1}(d\mu_0,\cos\Theta)\\&=\frac{(-1)^{[(d-1)/2]}}{2^{d-1}}\sum_{i=1}^{d}\frac{\cos\frac{\theta_i}{2}(\sin\theta_i)^{d-2}}{\prod_{j=1,i\neq j}^{d}(\cos\theta_i-\cos\theta_j)}\\&\quad\times\sum_{k=0}^{n-1}{}'\cos k\theta_{d+1}\sin\left(n-k-\frac{1}{2}\right)\theta_i\\&=\frac{(-1)^{[(d-1)/2]}}{2^{d}}\left[\sum_{i=1}^{d}\frac{\cos\frac{\theta_i}{2}(\sin\theta_i)^{d-1}\cos\left(n-\frac{1}{2}\right)\theta_i}{\prod_{j=1,i\neq j}^{d+1}(\cos\theta_i-\cos\theta_j)}\right.\\&\quad\left.-\cos\left(n-\frac{1}{2}\right)\theta_{d+1}\cos\frac{\theta_{d+1}}{2}\sum_{i=1}^{d}\frac{(\sin\theta_i)^{d-1}}{\prod_{j=1,i\neq j}^{d+1}(\cos\theta_i-\cos\theta_j)}\right].\end{aligned} \tag{4.2.8}$$

However, d is an odd integer, so the function $g(x)=(1-x^2)^{(d-1)/2}$ is a polynomial of degree $d-1$. Let $x_j=\cos\theta_j$ and $g[x_1, ..., x_d]$ denote the finite difference of g based on $x_1, ..., x_d$. We then have

$$\sum_{i=1}^{d+1} \frac{(\sin\theta_i)^{d-1}}{\prod_{j=1, i\neq j}^{d+1}(\cos\theta_i-\cos\theta_j)} = \sum_{i=1}^{d+1} \frac{(1-x^2)^{(d-1)/2}}{\prod_{j=1, i\neq j}^{d+1}(x_i-x_j)}$$
$$= g[x_1, ..., x_d] = 0,$$

where the last identity follows from the fact that the dth finite difference of a polynomial of degree $d-1$ is zero. In particular, we have

$$\sum_{i=1}^{d} \frac{(\sin\theta_i)^{d-1}}{\prod_{j=1, i\neq j}^{d+1}(\cos\theta_i-\cos\theta_j)} = -\frac{(\sin\theta_{d+1})^{d-1}}{\prod_{j=1}^{d}(\cos\theta_{d+1}-\cos\theta_j)}.$$

Substituting this formula into (4.2.8) and using the fact that $(-1)^{[(d-1)/2]}=(-1)^{[d/2]}$ for odd d, we conclude the proof for the case of odd d. The case of even d is proved similarly with $g(x)=(1-x^2)^{(d-2)/2}(1+x)$. ∎

We remark that this formula gives a compact formula for the reproducing kernel of the measure μ_0. Such a formula makes the investigation of the partial sums of Chebyshev orthogonal expansion in several variables possible. We intend to explore this formula in another place. For the application in the following section, we need

LEMMA 4.2.4. *For* $\Theta\in[0,\pi]^d$,

$$|L_n(\cos\Theta)| \leqslant \min\left\{n^d, \frac{1}{2^d}\sum_{i=1}^{d} \frac{1}{\sin\frac{\theta_i+\theta_{i'}}{2}\prod_{j=1, i\neq j}^{d}\left(\sin\frac{\theta_i-\theta_j}{2}\right)}\right\} \qquad (4.2.9)$$

where i' can be taken as either $i+1$ or $i-1$. Moreover, for $d=2$ and $\Theta\in[0,\pi]^2$, we have

$$|L_n(\cos\Theta)| \leqslant \frac{1}{2}\min\left\{n, \frac{1}{\sin\frac{\theta_1-\theta_2}{2}}\right\}\min\left\{n, \frac{1}{\sin\frac{\theta_1+\theta_2}{2}}\right\}. \qquad (4.2.10)$$

Proof. From the definition of L_n, we clearly have

$$|L_n(\cos\Theta)| \leqslant \sum_{|\mathbf{k}|\leqslant n-1} 1 = \binom{n+d-1}{d} \leqslant n^d$$

where the last inequality follows from (4.1.9). From (4.2.5) we have

$$|L_n(\cos\Theta)| \leqslant \frac{1}{2^{d-1}} \sum_{i=1}^{d} \frac{(\sin\theta_i)^{d-2}}{\prod_{j=1, i\neq j}^{d} |\cos\theta_i - \cos\theta_j|}.$$

The denominator can be rewritten by the use of the trigonometric identity

$$\cos\theta_i - \cos\theta_j = 2\sin\frac{\theta_i+\theta_j}{2}\sin\frac{\theta_j-\theta_i}{2}.$$

Since $\theta_j \in [0, \pi]$, it is easy to verify that

$$\sin\theta_i \leqslant 2\left|\sin\frac{\theta_i+\theta_j}{2}\right|$$

holds. Therefore, we can replace the $\sin\theta_i$ in the numerator by $\sin(\theta_i+\theta_j)/2$ in the denominator, which leads to the desired estimate (4.2.9). If $d=2$, then we obtain from (4.2.7) and elementary trigonometric identity that

$$L_{n,2}(\cos\Theta)$$
$$= \frac{1}{4}\left[\frac{\cos n\theta_1 - \cos n\theta_2}{\cos\theta_1 - \cos\theta_2} + \frac{\cos(n-1)\theta_1 - \cos(n-1)\theta_2}{\cos\theta_1 - \cos\theta_2}\right]$$
$$= \frac{1}{4}\left[\frac{\sin n\frac{\theta_1-\theta_2}{2}\sin n\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2}} + \frac{\sin(n-1)\frac{\theta_1-\theta_2}{2}\sin(n-1)\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2}}\right],$$

from which the estimate (4.2.10) follows readily. ∎

4.3. *The Asymptotics of the Christoffel Functions*

We extend the theorem in [2, p. 175] for orthogonal polynomials of one variable to that of two variables and discuss what causes difficulties for $d>2$. Our proof follows the method in [2], but the results we obtain are stronger than those of [2], see the remark at the end of the section.

THEOREM 4.3.1. *Let* $\mu = W(\mathbf{x})\,d\mathbf{x} \in \mathcal{M}([-1,1]^2)$ *and* $\mu_0 = W_0(\mathbf{x})\,d\mathbf{x}$. *Let the function* U *be defined by* $U(\mathbf{x}) = W(\mathbf{x})\,W_0^{-1}(\mathbf{x})$, $\mathbf{x}\in[-1,1]^2$ *and* $U(\mathbf{x})=0$ *outside* $[-1,1]^2$. *Suppose for* $\mathbf{h}$ *such that* $|\mathbf{h}|_2$ *sufficiently small,*

$$\int_{[0,\pi]^2}\left|\frac{U(\cos(\Theta+\mathbf{h}))}{U(\cos\Theta)} - 1\right| d\Theta \leqslant M\log^{-\beta}\frac{1}{|\mathbf{h}|_2} \qquad (4.3.1)$$

for some $\beta > 1$. *Then for almost every* $\mathbf{x} \in [-1, 1]^2$

$$\begin{aligned} 2^{-2}[U(x)]^{-1} &\leqslant \liminf_{n \to \infty} \left[\binom{n+1}{2} \right]^{-1} \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \\ &\leqslant \limsup_{n \to \infty} \left[\binom{n+1}{2} \right]^{-1} \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \\ &\leqslant 2^2[U(\mathbf{x})]^{-1}. \end{aligned} \tag{4.3.2}$$

Proof. We shall start the proof for $\mu \in \mathcal{M}(\mathbb{R}^d)$ and restrict to $d=2$ at the point when we have to use the special estimate (4.2.10). We shall discuss the difficulties for $d>2$ after we complete the proof of this theorem. From (2.7), we have

$$\begin{aligned} \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) &= [\Lambda_n(d\mu, \mathbf{x})]^{-1} \\ &= \max_{P \in \Pi_{n-1}^d} \frac{P^2(\mathbf{x})}{\int_{[-1,1]^d} P^2(\mathbf{y})\, d\mu(\mathbf{y})} \\ &\geqslant \frac{\mathbf{K}_n^2(d\mu_0, \mathbf{x}, \mathbf{x})}{\int_{[-1,1]^d} \mathbf{K}_n^2(d\mu_0, \mathbf{x}, \mathbf{y})\, d\mu(\mathbf{y})}. \end{aligned}$$

By the reproducing property of $\mathbf{K}_n(d\mu_0)$, we write the denominator of the last integral as

$$\begin{aligned} &\int_{[-1,1]^d} \mathbf{K}_n^2(d\mu_0, \mathbf{x}, \mathbf{y})\, d\mu(\mathbf{y}) \\ &\quad = U(\mathbf{x}) \int_{[-1,1]^d} \mathbf{K}_n^2(d\mu_0, \mathbf{x}, \mathbf{y})\, d\mu_0(\mathbf{y}) \\ &\qquad + \int_{[-1,1]^d} \mathbf{K}_n^2(d\mu_0, \mathbf{x}, \mathbf{y})[U(\mathbf{y}) - U(\mathbf{x})]\, d\mu_0(\mathbf{y}) \\ &\quad = U(\mathbf{x})\, \mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x}) + U(\mathbf{x})\, J_n(\mathbf{x}) \end{aligned}$$

where

$$J_n(\mathbf{x}) = \int_{[-1,1]^d} \mathbf{K}_n^2(d\mu_0, \mathbf{x}, \mathbf{y}) \frac{U(\mathbf{y}) - U(\mathbf{x})}{U(\mathbf{x})}\, d\mu_0(\mathbf{y}).$$

Thus, we obtain

$$U(\mathbf{x})\, \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \geqslant \frac{\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x})}{1 + [\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x})]^{-1} J_n(\mathbf{x})}.$$

Using the inequality

$$\frac{a}{1+a^{-1}b} = \frac{a^2}{a+b} \geqslant a-b$$

which holds for all real a, b such that $a+b>0$, we obtain

$$U(\mathbf{x})\,\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \geqslant \mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{x}) - J_n(\mathbf{x}).$$

Therefore, let $N_n = \binom{n+d-1}{d}$, we have by (4.1.5),

$$U(\mathbf{x})\,\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \geqslant N_n - |R_n(\mathbf{x})| - J_n(\mathbf{x}),$$

which implies

$$1 - \frac{1}{N}\,U(\mathbf{x})\,\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \leqslant \frac{1}{N_n}\,|R_n(\mathbf{x})| + \frac{1}{N}\,|J_n(\mathbf{x})|. \tag{4.3.3}$$

Let c^+ denote the positive part of a real number c, i.e., $c^+ = c$ if $c>0$, and $c^+ = 0$ if $c \leqslant 0$. Then, since $N_n = \dim \prod_{n-1}^d$, and

$$\begin{aligned}
&\int_{[-1,1]^d} \left[1 - \frac{1}{N_n}\,U(\mathbf{x})\,\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x})\right] d\mu_0 \\
&\quad = \int_{[-1,1]^d} d\mu_0 - \frac{1}{N_n}\int_{[-1,1]^d} \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x})\,d\mu = 1 - 1 = 0,
\end{aligned}$$

we obtain from (4.3.3) and (4.1.7)

$$\begin{aligned}
G_n &:= \int_{[-1,1]^d} \left|1 - \frac{1}{N_n}\,U(\mathbf{x})\,\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x})\right| d\mu_0 \\
&= 2\int_{[-1,1]^d} \left[1 - \frac{1}{N_n}\,U(\mathbf{x})\,\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x})\right]^+ d\mu_0 \\
&\leqslant \frac{2}{N_n}\int_{[-1,1]^d} |R_n(\mathbf{x})|\,d\mu_0 + \frac{2}{N_n}\int_{[-1,1]^d} |J_n(\mathbf{x})|\,d\mu_0 \\
&\leqslant c\,\frac{\log n}{n} + \frac{2}{N_n}\int_{[-1,1]^d} |J_n(\mathbf{x})|\,d\mu_0.
\end{aligned} \tag{4.3.4}$$

To estimate the last integral, we first estimate the integrand $J_n(\mathbf{x})$. Changing variables from $x_i = \cos\theta_i$ and $y_i = \cos\phi_i$, $1 \leqslant i \leqslant d$, we use the formulae in Lemma 4.2.1 and Lemma 4.2.4. By (4.2.1) and Cauchy's inequality we have readily

$$|\mathbf{K}_n(d\mu_0, \mathbf{x}, \mathbf{y})|^2 \leqslant 2^d \sum_{\varepsilon \in \{-1,1\}^d} |L_n(\cos(\Phi + \varepsilon\Theta))|^2.$$

Therefore,

$$|J_n(\mathbf{x})| = \left| \int_{[0,\pi]^d} \mathbf{K}_n^2(d\mu_0, \cos\Theta, \cos\Phi) \frac{U(\cos\Phi) - U(\cos\Theta)}{U(\cos\Theta)} \, d\Phi \right|$$

$$\leqslant 2^d \sum_{\varepsilon \in \{-1,1\}^d} \int_{[0,\pi]^d} |L_n(\cos(\Phi + \varepsilon\Theta)|^2 \times \left| \frac{U(\cos\Phi) - U(\cos\Theta)}{U(\cos\Theta)} \right| d\Phi.$$

Integrating against Θ and changing variables $\Phi = \mathbf{h} - \varepsilon\Theta$, we obtain

$$\int_{[-1,1]^d} |J_n(\mathbf{x})| \, d\mu_0 \leqslant 2^d \int_{[0,\pi]^d} \sum_{\varepsilon \in \{-1,1\}^d} \int_{[-\pi,\pi]^d} |L_n(\cos\mathbf{h})|^2 \times \left| \frac{U(\cos(\mathbf{h} - \varepsilon\Theta)) - U(\cos\Theta)}{U(\cos\Theta)} \right| d\mathbf{h} \, d\Theta$$

$$\leqslant 2^{2d} \int_{[0,\pi]^d} |L_n(\cos\mathbf{h})|^2 \times \int_{[-\pi,\pi]^d} \left| \frac{U(\cos(\Theta + \mathbf{h})) - U(\cos\Theta)}{U(\cos\Theta)} \right| d\Theta \, d\mathbf{h}.$$

We define $g(\mathbf{h}) = \log^{-\beta}(1/|\mathbf{h}|_2)$ for $|\mathbf{h}|_2 \leqslant 1$ and $g(\mathbf{h})$ bounded by 1 for $|\mathbf{h}|_2 > 2$. Then we can use (4.3.1) in the last integral to obtain the estimate

$$\int_{[-1,1]^d} |J_n(\mathbf{x})| \, d\mu_0 \leqslant 2^{2d} M \int_{[0,\pi]^d} |L_n(\cos\mathbf{h})|^2 \, g(\mathbf{h}) \, d\mathbf{h}. \tag{4.3.5}$$

It is at this point we need to restrict ourself to $d = 2$ and use the estimate (4.2.10) to conclude that

$$\int_{[-1,1]^2} |J_n(\mathbf{x})| \, d\mu_0 \leqslant 2^3 M \int_{[0,\pi]^2} \min\left\{ n^2, \frac{1}{\sin^2 \frac{h_1 - h_2}{2}} \right\} \times \min\left\{ n^2, \frac{1}{\sin^2 \frac{h_1 + h_2}{2}} \right\} g(\mathbf{h}) \, d\mathbf{h}.$$

For this integral we change variables from $\mathbf{h}$ to $\mathbf{u}$ with

$$u_1 = h_1 + h_2, \qquad \text{and} \qquad u_2 = h_1 - h_2.$$

We can use the inequality $|\sin u_i| \geqslant (2/\pi)|u_i|$, $0 \leqslant u_i \leqslant \pi/2$, to replace $\sin u_i$ in the integrand by u_i. Notice that $\log^{-\beta}(1/t)$ is increasing for $0 < t \leqslant 1$, and $|\mathbf{h}|_2 \leqslant \sqrt{2}\,|\mathbf{u}|_2$, we can replace $g(\mathbf{h})$ by $g(\mathbf{u})$. Enlarging the domain of the integral we obtain

$$\int_{[-1,1]^2} |J_n(\mathbf{x})|\, d\mu_0 \leqslant c \int_{\mathbb{R}^2} \min\left\{n^2, \frac{1}{u_1^2}\right\} \min\left\{n^2, \frac{1}{u_2^2}\right\} g(\mathbf{u})\, d\mathbf{u}.$$

Splitting the integral over $\mathbb{R}^2$ into the integrals over $\{|u| \leqslant 1/n\}$, $\{u_1 \leqslant 1/n, u_2 \geqslant 1/n\}$, and $\{u_1 \geqslant 1/n,\ u_2 \geqslant 1/n\}$, we can estimate the last integral and conclude that

$$\int_{[-1,1]^2} |J_n(\mathbf{x})|\, d\mu_0 \leqslant cn^2 \log^{-\beta} n.$$

Therefore, by (4.3.4) we have established

$$G_n \leqslant c\left(\frac{\log n}{n} + \frac{1}{(\log n)^\beta}\right) \leqslant c\,\frac{1}{(\log n)^\beta}.$$

Since $\beta > 1$, we have

$$\sum_{r=1}^{\infty} G_{2^r} \leqslant c \sum_{r=1}^{\infty} r^{-\beta} < \infty.$$

Therefore, by Levi's theorem (cf. [4, p. 305]), we have

$$\sum_{r=0}^{\infty} \left| 1 - \frac{1}{N_{2^r}} U(\mathbf{x})\, \mathbf{K}_{2^r}(d\mu, \mathbf{x}, \mathbf{x}) \right| < \infty$$

for almost every $\mathbf{x} \in [-1, 1]^2$. In particular, the terms in the summation converge to zero,

$$\lim_{r \to \infty} \left| 1 - \frac{1}{N_{2^r}} U(\mathbf{x})\, \mathbf{K}_{2^r}(d\mu, \mathbf{x}, \mathbf{x}) \right| = 0 \tag{4.3.6}$$

for almost every $\mathbf{x} \in [-1, 1]^2$. Moreover, we have from (4.3.6)

$$1 - \varepsilon(\mathbf{x}) \leqslant \frac{1}{N_{2^r}} U(\mathbf{x})\, \mathbf{K}_{2^r}(d\mu, \mathbf{x}, \mathbf{x}) \leqslant 1 + \varepsilon(\mathbf{x}),$$

where $\varepsilon(\mathbf{x}) \to 0$ as $r \to \infty$. Since $\mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x})$ is a nondecreasing function of n, by choosing r such that $2^{r-1} \leqslant n \leqslant 2^r$, we obtain

$$\begin{aligned} U(\mathbf{x})\, \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) &\leqslant U(\mathbf{x})\, \mathbf{K}_{2^r}(d\mu, \mathbf{x}, \mathbf{x}) \leqslant N_{2^r}(1 + \varepsilon(\mathbf{x})) \\ &\leqslant N_{2n}(1 + \varepsilon(\mathbf{x})). \end{aligned}$$

Since

$$N_{2n}=\binom{2n+1}{2}=2^2N_n(1+O(n^{-1})),$$

we obtain

$$U(\mathbf{x})\,\mathbf{K}_n(d\mu,\mathbf{x},\mathbf{x})\leqslant 2^2N_n(1+\varepsilon(\mathbf{x})+O(n^{-1}))$$

which implies

$$\limsup_{n\to\infty}\frac{1}{N_n}\,U(\mathbf{x})\,\mathbf{K}_n(d\mu,\mathbf{x},\mathbf{x})\leqslant 2^2$$

for almost every $\mathbf{x}\in[-1,1]^2$. From this inequality follows the third inequality in (4.3.2). Using

$$U(\mathbf{x})\,\mathbf{K}_n(d\mu,\mathbf{x},\mathbf{x})\geqslant U(\mathbf{x})\,\mathbf{K}_{2^r-1}(d\mu,\mathbf{x},\mathbf{x})\geqslant U(\mathbf{x})\,\mathbf{K}_{n/2}(d\mu,\mathbf{x},\mathbf{x}),$$

the first inequality in (4.3.2) follows similarly. ∎

For $d>2$, one would naturally try to use (4.2.9) in Lemma 4.2.4 to estimate the integral in (4.3.5). By using an elementary inequality

$$\min\left\{a,\sum_{i=1}^{d}c_i\right\}\leqslant d\sum_{i=1}^{d}\min\{a,c_i\},\qquad a\geqslant 0,\quad c_i\geqslant 0,$$

and the estimate (4.2.9), we obtain

$$\int_{[-1,1]^d}|J_n(\mathbf{x})|\,d\mu_0$$

$$\leqslant d2^{2d}M\sum_{i=1}^{d}\int_{[0,\pi]^d}\min\left\{n^{2d},\frac{1}{2^{2d}}\frac{1}{\sin^2\dfrac{h_i+h_{i'}}{2}\displaystyle\prod_{j=1,\,i\neq j}^{d}\left(\sin\frac{h_i-h_j}{2}\right)^2}\right\}g(\mathbf{h})\,d\mathbf{h}.$$

We only need to consider one of the integrals, say, the first one. For this integral we change variables from $\mathbf{h}$ to $\mathbf{u}$ with

$$u_1=h_1+h_2,\qquad u_2=h_1-h_2,\qquad u_3=h_1-h_3,\ldots,u_d=h_1-h_d,$$

it is readily seen that the Jacobian is equal to 2. Enlarging the domain of the integral and the integrand as in the proof of the theorem, we obtain the estimate

$$\int_{[-1,1]^d}|J_n(\mathbf{x})|\,d\mu_0\leqslant c\int_{\mathbb{R}^d}\min\left\{n^{2d},\frac{1}{\prod_{i=1}^{d}u_i^2}\right\}g(\mathbf{u})\,d\mathbf{u}.$$

However, the last integral is not bounded by the desired bound $cn^d \log^{-\beta} n$. Actually, the region where

$$\prod_{i=1}^{d} u_i^{-2} \geqslant n^{2d}$$

is bounded by $u_1 \cdots u_d \leqslant n^{-d}$ which, for $d=2$, is a hyperbolic curve. It is not hard to show that the integral of $g(\mathbf{u})$ over this region is bounded below by a constant times n^{-d}; indeed, consider a subset of the region by restricting $1/2 \leqslant u_1 \leqslant 1$, then on this subset $g(\mathbf{u}) \geqslant g(1/2, n^{-2}, ..., n^{-2}) \geqslant \log^{-\beta} 2$ and the area of the subset is bounded below by a constant times n^{-d}. Therefore, the last integral is bounded below by a constant times n^d and cannot be bounded by $cn^d \log^{-\beta} n$ from above. The problem is that the region is not centered, or localized, around the origin, while as a radial function $g(\mathbf{u})$ is centered around the origin. This indicates that the estimate (4.2.9) is not strong enough, an estimate of a new type is called for. Incidentally, the problem seems to be more serious due to the following fact: the l^1 partial sums of Fourier series on $\mathbb{T}^d$, which means that the partial sums are taken over multi-integers inside the l^1 ball $|\mathbf{k}| \leqslant n-1$, do not have the localization property for $d \geqslant 3$ ([3, p. 751]), while the function $L_n(\cos \Theta)$ is exactly the Dirichlet kernel for the nth l^1 partial sum.

The results in this theorem can be stated in terms of the Christoffel function.

COROLLARY 4.3.2. *Let the assumptions be the same as in Theorem* 4.3.1. *Then for almost every* $\mathbf{x} \in [-1, 1]^2$

$$\begin{aligned}\frac{\pi^2}{2^2} W(\mathbf{x}) \sqrt{1-x_1^2}\sqrt{1-x_2^2} &\leqslant \liminf_{n\to\infty} \binom{n+1}{2} \Lambda_n(d\mu, \mathbf{x}) \\ &\leqslant \limsup_{n\to\infty} \binom{n+1}{2} \Lambda_n(d\mu, \mathbf{x}) \\ &\leqslant 2^2\pi^2 W(\mathbf{x}) \sqrt{1-x_1^2}\sqrt{1-x_2^2}, \qquad (4.3.7)\end{aligned}$$

moreover,

$$\lim_{r\to\infty} \binom{2^r+1}{2} \Lambda_{2^r}(d\mu, \mathbf{x}) = \pi^2 W(\mathbf{x}) \sqrt{1-x_1^2}\sqrt{1-x_2^2}. \qquad (4.3.8)$$

The last relation (4.3.8) follows from (4.3.6). This result should be compared to (1.1) for orthogonal polynomials of one variable; while (1.1) is much stronger, as mentioned in introduction, its proof relies on Szegő's theory.

We remark that the proof of this theorem follows the method in [2, Theorem 6.7, p. 175] for orthogonal polynomials of one variable, however, our results when taking $d=1$ are stronger than those of [2, Theorem 6.7]. Actually, the result of [2, Theorem 6.7] could have been stated as the limit relations like in our Theorem 4.3.1, except that there is a small error in one formula of [2], namely, the right hand side of the limit equation on the middle of [2, p. 180] should have been $1/\pi w(\Theta)$.

5. Summability of Multivariate Orthogonal Polynomials

In this section we study the summability of the partial sums of multivariate orthogonal polynomials. For $f \in L^2_{d\mu}$, let $E_n(d\mu, f)_2$ be the error of best $L^2_{d\mu}$ approximation from Π^d_n, i.e.,

$$E_n(d\mu, f)_2 = \inf_{P \in \Pi^d_n} \|f - P\|_{d\mu, 2}.$$

We assume that $E_n(d\mu, f)_2 \to 0$ as $n \to \infty$ and denote by $\mathcal{M}_0(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ the set of measures that satisfy this assumption. If μ has compact support, then $\mu \in \mathcal{M}_0(\mathbb{R}^d)$. For $f \in C(\Omega)$, $\Omega \subset \mathbb{R}^d$, we denote by $\|f\|_{\infty, \Omega}$ the uniform norm of f on Ω, and $E_n(f)_{\infty, \Omega}$ the error of best uniform approximation from Π^d_n,

$$E_n(f)_{\infty, \Omega} = \inf_{P \in \Pi^d_n} \|f - P\|_{\infty, \Omega}.$$

First we consider the convergence of the partial sum operator $S_n(d\mu, f)$.

LEMMA 5.1. *Let $\mu \in \mathcal{M}_0(\mathbb{R}^d)$. If for $\mathbf{x} \in \mathbb{R}^d$*

$$\sum_{m=0}^{\infty} \frac{E_{2^m}(d\mu, f)_2}{\sqrt{\Lambda_{2^m+1}(d\mu, \mathbf{x})}} < \infty, \tag{5.1}$$

then $S_n(d\mu, f)$ converges at the point $\mathbf{x}$. If (5.1) holds uniformly on a set Δ, then $S_n(d\mu)$ converges uniformly on Δ.

Proof. Since $S_n(d\mu, f)$ is the best $L^2_{d\mu}$ approximation of $f \in L^2_{d\mu}$ from Π^d_{n-1}, and $E_n(d\mu, f)_2$ tends to zero as $n \to \infty$, we have by (2.4), the Parseval identity (2.5), and the orthogonality of $\mathbb{P}_n(d\mu)$ that

$$\begin{aligned} E_n(d\mu, f)_2^2 &= \int_{\mathbb{R}^d} [f - S_n(d\mu, f)]^2 \, d\mu \\ &= \int_{\mathbb{R}^d} \left[\sum_{k=n+1}^{\infty} \mathbf{a}_k^T(d\mu, f) \, \mathbb{P}_k(d\mu) \right]^2 d\mu \\ &= \sum_{k=n+1}^{\infty} |\mathbf{a}_k(d\mu, f)|_2^2. \end{aligned}$$

Therefore, using the Cauchy–Schwarz inequality, we obtain

$$\left(\sum_{k=2^m+1}^{2^{m+1}} |\mathbf{a}_k^T(d\mu,f)\,\mathbb{P}_k(d\mu,\mathbf{x})|\right)^2 \leqslant \sum_{k=2^m+1}^{2^{m+1}} |\mathbf{a}_k^T(d\mu,f)|_2^2 \sum_{k=2^m+1}^{2^{m+1}} |\mathbb{P}_k(d\mu,\mathbf{x})|_2^2$$
$$\leqslant \sum_{k=0}^{2^{m+1}} |\mathbb{P}_k(d\mu,\mathbf{x})|_2^2 \sum_{k=2^m+1}^{\infty} |\mathbf{a}_k^T(d\mu,f)|_2^2$$
$$= [E_{2^m}(d\mu,f)_2]^2\,[\Lambda_{2^{m+1}}(d\mu,\mathbf{x})]^{-1}.$$

Taking the square root and summing over m proves the desired result. ∎

Lemma 5.2. *Let $\mu \in \mathcal{M}_0(\mathbb{R}^d)$. If*

$$[n^d \Lambda_n(d\mu,\mathbf{x})]^{-1} \leqslant M^2 \tag{5.2}$$

holds uniformly on a set Δ, then $S_n(d\mu,f)$ is uniformly and absolutely convergent on Δ for every $f \in L^2_{d\mu}$ such that

$$\sum_{k=1}^{\infty} E_k(d\mu,f)_2\, k^{(d-2)/2} < \infty. \tag{5.3}$$

Proof. Since $E_n(d\mu,f)_2$ is nonincreasing, we have

$$\frac{E_{2^m}(d\mu,f)_2}{\sqrt{\Lambda_{2^{m+1}}(d\mu,\mathbf{x})}} \leqslant M 2^{(m+1)d/2} \frac{1}{2^{m-1}} \sum_{k=2^{m-1}+1}^{2^m} E_k(d\mu,f)_2$$
$$= M 2^{(d/2)+1} 2^{m((d-2)/2)} \sum_{k=2^{m-1}+1}^{2^m} E_k(d\mu,f)_2$$
$$\leqslant c \sum_{k=2^{m-1}+1}^{2^m} E_k(d\mu,f)_2\, k^{(d-2)/2}$$

Summing over m, the desired result follows from Lemma 5.1. ∎

These results together with their proofs are almost straightforward extensions of the ones for $d=1$ (cf. [2, p. 139]). For $d=1$ and $\operatorname{supp}\mu = [-1,1]$, condition (5.3) is satisfied, for example, for $f \in \operatorname{Lip}\beta$, $\beta > 1/2$. But for $d > 1$, much more on f is required. As an example we formulate the following:

Corollary 5.3. *Let $\mu \in \mathcal{M}_0(\mathbb{R}^d)$, $\operatorname{supp}\mu = [-1,1]^d$, and suppose that (5.2) holds uniformly on $\Delta \subset [-1,1]^d$. Suppose $f \in C^{[d/2]}([-1,1]^d)$ and each of its $[d/2]$th derivatives satisfies*

$$|D^{[d/2]}f(\mathbf{x}) - D^{[d/2]}f(\mathbf{y})| \leqslant ch^{\beta}, \qquad |\mathbf{x}-\mathbf{y}|_2 \leqslant h,$$

where for odd d, $\beta > 1/2$, and for even d, $\beta > 0$. Then $S_n(d\mu, f)$ converges uniformly and absolutely to f on Δ.

Proof. By [8, p. 90], for f satisfying the above assumptions, there exists $P \in \Pi_n^d$ such that

$$\|f - P\|_\infty \leqslant cn^{-([d/2]+\beta)},$$

which implies

$$E_n(d\mu, f)_2 \leqslant cn^{-([d/2]+\beta)}.$$

Using this estimate, our assumption on β implies that (5.3) holds. ▌

The fact that for $d > 1$ we need to assume conditions on the higher order derivatives of f does not seem to be accidental. It comes from the assumption that $[n^d \Lambda_n(d\mu, \mathbf{x})]^{-1}$ is bounded, which seems to be quite natural from the fact that for $d = 1$ the function $S_n(d\alpha, f)$ is a sum of n terms, but for $d > 1$ the function $S_n(d\mu, f)$ is a sum of $O(n^d)$ terms. Our consideration below of the first Cesáro mean of the orthogonal series will further illustrate this phenomenon. In the following, we write $\mathbf{y} = (y_1, \cdots, y_d)$ and $\mathbf{t} = (t_1, ..., t_d)$.

THEOREM 5.4. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure, and $\Omega := \operatorname{supp} \mu$ be a compact subset $\subseteq [-1, 1]^d$. We define $W(\mathbf{x}) = 0$ for $\mathbf{x}$ outside Ω. Let $\mathbf{x} = (x_1, ..., x_d) \in [-1, 1]^d$ be fixed. Suppose μ satisfies*

$$\int_{y_i}^{x_i} W(\mathbf{t})\, dt_i \leqslant M |x_i - y_i|, \qquad M < \infty, \tag{5.4}$$

for some i, $1 \leqslant i \leqslant d$. Then for $f \in C(\Omega)$,

$$\begin{aligned} &\frac{1}{n} \sum_{k=0}^{n-1} |S_k(d\mu, f, \mathbf{x})| \\ &\quad \leqslant c \|f\|_{\infty, \Omega} [n\Lambda_n(d\mu, \mathbf{x})]^{-1/2} \left(1 + \frac{1}{n(1 - x_i^2)}\right)^{1/2}, \qquad x \in \Omega. \end{aligned} \tag{5.5}$$

Moreover, if Δ is a subset of Ω for which $\min_{\mathbf{x} \in \Delta}\{1 - x_i^2\} \geqslant \delta > 0$, where i is the index in (5.4), then (5.5) holds uniformly on Δ.

Proof. For $\mathbf{x} \in \Omega$, let $\Delta_{n,i} = \{\mathbf{y} \in \Omega : |y_i - x_i| \leqslant n^{-1}\}$, where i is the index in (5.4). For $k \in \mathbb{N}_0$ we write

$$S_k(d\mu, f, \mathbf{x}) = S_{k,1}(d\mu, f, \mathbf{x}) + S_{k,2}(d\mu, f, \mathbf{x}),$$

where

$$S_{k,1}(d\mu, f, \mathbf{x}) = \int_{\Delta_{n,i}} \mathbf{K}_k(d\mu, \mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mu(\mathbf{y}),$$

and

$$S_{k,2}(d\mu, f, \mathbf{x}) = \int_{\Omega\setminus\Delta_{n,i}} \mathbf{K}_k(d\mu, \mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mu(\mathbf{y}).$$

For a set $\Delta \subset \mathbb{R}^d$, denote by $|\Delta|$ its d-dimensional Lebesgue measure. Then, since $|\Delta_{n,i}| \leqslant 2^d/n$, we have by the Cauchy-Schwarz inequality

$$\begin{aligned}
|S_{k,1}(d\mu, f, \mathbf{x})|^2 &\leqslant \int_{\Delta_{n,i}} [\mathbf{K}_k(d\mu, \mathbf{x}, \mathbf{y})]^2\, d\mu(\mathbf{y}) \int_{\Delta_{n,i}} [f(\mathbf{y})]^2\, d\mu(\mathbf{y}) \\
&\leqslant \mathbf{K}_k(d\mu, \mathbf{x}, \mathbf{x}) \int_{\Delta_{n,i}} [f(\mathbf{y})]^2\, d\mu(\mathbf{y}) \\
&\leqslant \mathbf{K}_n(d\mu, \mathbf{x}, \mathbf{x}) \int_{\Delta_{n,i}} [f(\mathbf{y})]^2\, d\mu(\mathbf{y}) \\
&\leqslant 2^d \|f\|^2_{\infty,\Omega} [\Lambda_n(d\mu, \mathbf{x})]^{-1} n^{-1}.
\end{aligned}$$

Therefore,

$$\frac{1}{n} \sum_{k=0}^{n-1} |S_{k,1}(d\mu, \mathbf{x})| \leqslant 2^{d/2} \|f\|_{\infty,\Omega} [n\Lambda_n(d\mu, \mathbf{x})]^{-1/2}. \tag{5.6}$$

By the Christoffel-Darboux formula (2.3) for the index i in (5.4), we can write

$$\begin{aligned}
S_{k,2}(d\mu, f, \mathbf{x}) &= \mathbb{P}_k^T(d\mu, \mathbf{x})\, A_{i,k-1}^T \mathbf{a}_{k-1}(d\mu, F) \\
&\quad - \mathbb{P}_{k-1}^T(d\mu, \mathbf{x})\, A_{i,k-1} \mathbf{a}_k(d\mu, F)
\end{aligned}$$

where $\mathbf{a}_k(d\mu, F)$ is the Fourier coefficient of F, and

$$F(\mathbf{y}) = \begin{cases} \dfrac{f(\mathbf{y})}{x_i - y_i} & \text{if } \mathbf{y} \in \Omega\setminus\Delta_{n,i} \\ 0 & \text{if } \mathbf{y} \in \Delta_{n,i}. \end{cases}$$

Since $f \in L^2_{d\mu}$, we have $F \in L^2_{d\mu}$ as well. By Parseval's identity, we then have

$$\sum_{k=0}^{n-1} |\mathbf{a}_k(d\mu, F)|_2^2 \leqslant \sum_{k=0}^{\infty} |\mathbf{a}_k(d\mu, F)|_2^2 = \int_{\Omega} |F(\mathbf{y})|^2\, d\mu(\mathbf{y}).$$

Since Ω is compact, we have by (2.2) that $\sup_{k\geqslant 0}|A_{k,i}|_2<\infty$. Therefore, by the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}|S_{k,2}(d\mu,\mathbf{x})| &\leqslant 2\sup_{k\geqslant 0}|A_{k,i}|_2\sum_{k=0}^{n-1}(|\mathbb{P}_k(d\mu,\mathbf{x})|_2\,|\mathbf{a}_{k-1}(d\mu)|_2\\ &\quad+|\mathbb{P}_{k-1}(d\mu,\mathbf{x})|_2\,|\mathbf{a}_k(d\mu)|_2)\\ &\leqslant 2\sup_{k\geqslant 0}|A_{k,i}|_2\left(\sum_{k=0}^{n-1}|\mathbb{P}_k(d\mu,\mathbf{x})|_2^2\right)^{1/2}\left(\sum_{k=0}^{n-1}|\mathbf{a}_k(d\mu,F)|_2^2\right)^{1/2}\\ &\leqslant 2\sup_{k\geqslant 0}|A_{k,i}|_2\,[\Lambda_n(d\mu,\mathbf{x})]^{-1/2}\left(\int_\Omega|F(\mathbf{y})|^2\,d\mu(\mathbf{y})\right)^{1/2}.\end{aligned}\tag{5.7}$$

Since $d\mu=W d\mathbf{x}$, and f is continuous on Ω, we have

$$\int_\Omega|F(\mathbf{y})|^2\,d\mu(\mathbf{y})\leqslant\|f\|_{\infty,\Omega}^2\int_{\Omega\setminus\Delta_{n,i}}\frac{1}{|x_i-y_i|^2}\,W(\mathbf{y})\,d\mathbf{y}.$$

For $\mathbf{y}\in\mathbb{R}^d$ we write $\mathbf{y}'=(y_1,\ldots,y_{i-1},y_{i+1},y_d)\in\mathbb{R}^{d-1}$. Let χ_Ω be the characteristic function of Ω. Then the last integral is less than or equal to

$$\begin{aligned}&\int_{[-1,1]^{d-1}}\int_{|x_i-y_i|\geqslant 1/n}\frac{\chi_\Omega(\mathbf{y})}{(x_i-y_i)^2}\,W(\mathbf{y})\,dy_i\,d\mathbf{y}'\\ &\quad=\int_{[-1,1]^{d-1}}\left(\int_{-1}^{x_i-(1/n)}+\int_{x_i+(1/n)}^{1}\right)\frac{\chi_\Omega(\mathbf{y})}{(x_i-y_i)^2}\,W(\mathbf{y})\,dy_i\,d\mathbf{y}'.\end{aligned}$$

Let $\mathbf{y}(t)=(y_1,\ldots,y_{i-1},t,y_{i+1},\ldots,y_d)$. By integration by parts, the first integral between brackets is equal to

$$\begin{aligned}&\frac{1}{(x_i-y_i)^2}\int_{x_i}^{y_i}\chi_\Omega(\mathbf{y}(t))\,W(\mathbf{y}(t))\,dt\,\Bigg|_{y_i=-1}^{x_i-(1/n)}+2\int_{-1}^{x_i-(1/n)}\\ &\quad\times\frac{1}{(x_i-y_i)^3}\int_{y_i}^{x_i}\chi_\Omega(\mathbf{y}(t))\,W(\mathbf{y}(t))\,dt\,dy_i\\ &\leqslant\frac{1}{(1+x_i)^2}\int_{-1}^{x_i}\chi_\Omega(\mathbf{y}(t))\,W(\mathbf{y}(t))\,dt+2\int_{-1}^{x_i-(1/n)}\frac{1}{(x_i-y_i)^3}\\ &\quad\times\int_{y_i}^{x_i}\chi_\Omega(\mathbf{y}(t))\,W(\mathbf{y}(t))\,dt\,dy_i.\end{aligned}$$

Therefore, by Fubini's theorem and the condition (5.4) we obtain

$$\int_{[-1,1]^{d-1}} \int_{-1}^{x_i-(1/n)} \frac{\chi_\Omega(\mathbf{y})}{|x_i-y_i|^2} W(\mathbf{y})\, dy_i\, d\mathbf{y}'$$
$$\leqslant \frac{1}{(1+x_i)^2} \int_{[-1,1]^{d-1}} \int_{-1}^{x_i} \chi_\Omega(\mathbf{y}(t))\, W(\mathbf{y}(t))\, dt\, d\mathbf{y}'$$
$$+2\int_{-1}^{x_i-(1/n)} \int_{[-1,1]^{d-1}} \int_{y_i}^{x_i} \chi_\Omega(\mathbf{y}(t))\, W(\mathbf{y}(t))\, dt\, d\mathbf{y}' \frac{dy_i}{(x_i-y_i)^3}$$
$$\leqslant M\frac{1}{1+x_i}\int_{[-1,1]^{d-1}} \chi_\Omega\, d\mathbf{y}' + 2M\int_{-1}^{x_i-(1/n)}$$
$$\times \int_{[-1,1]^{d-1}} \chi_\Omega(\mathbf{y})\, d\mathbf{y}' \frac{dy_i}{(x_i-y_i)^2}$$
$$\leqslant \frac{M2^{d-1}}{1+x_i} + 2Mn\,|\Omega|.$$

Similarly, we have

$$\int_{[-1,1]^{d-1}} \int_{x_i+(1/n)}^{1} \frac{\chi_\Omega(\mathbf{y})}{|x_i-y_i|^2} W(\mathbf{y})\, dy_i\, d\mathbf{y}' \leqslant \frac{M}{1-x_i} + 2Mn|\Omega|.$$

Therefore, we have proved that

$$\int_\Omega |F(\mathbf{y})|^2\, d\mu(\mathbf{y}) \leqslant \|f\|^2_{\chi,\Omega} \left[\frac{M}{1-x_i} + \frac{M}{1+x_i} + 4Mn|\Omega|\right].$$

Hence, putting all estimates together, we obtain from (5.7) that

$$\frac{1}{n}\sum_{k=0}^{n-1} |S_{k,2}(d\mu,\mathbf{x})| \leqslant 4\,\|f\|_\chi \sup_{k\geqslant 0} |A_{k,i}|_2\, [\Lambda_n(d\mu,\mathbf{x})]^{-1/2}$$
$$\times\left(\frac{M}{1-x_i} + \frac{M}{1+x_i} + 4Mn|\Omega|\right)^{1/2} n^{-1}$$
$$\leqslant c[n\Lambda_n(d\mu,\mathbf{x})]^{-1/2}\left(1+\frac{1}{n(1-x_i^2)}\right)^{1/2}.$$

The desired result follows from the inequality and (5.6). ▮

We remark that by using the Christoffel–Darboux formula and the three-term relation repeatedly, we can show that

$$(x_1-y_1)\cdots(x_d-y_d)\,\mathbf{K}_n(\mathbf{x},\mathbf{y}) = \sum_{u,v=-d}^{d-1} \mathbb{P}^T_{n+u}(\mathbf{x})\, C_{u,v}\mathbb{P}_{n+v}(\mathbf{y})$$

where some $C_{u,v,n}$ may be zero and others are products of matrices $A_{k,i}$ and $B_{k,i}$, $n-d \leqslant k \leqslant n+d-1$. For μ that has compact support, $|C_{u,v,n}|_2$ are uniformly bounded. This formula could be used instead of the Christoffel–Darboux formula in the proof of the above theorem. Using $\Delta_n = \{\mathbf{y}: |\mathbf{y}-\mathbf{x}|_2 < n^{-1}\}$ instead of $\Delta_{n,i}$, we can show that the term $n^{-1} \sum S_{k,1}(d\mu, f, \mathbf{x})$ is bounded by $c[n^d A(d\mu, \mathbf{x})]^{-1}$. However, the estimate of the second term involving $S_{k,2}$ becomes worse than what we have above.

For $d=1$, a more delicate analysis is used in [2] to establish the strong $(C, 1)$ summability of orthogonal polynomials of one variable, which holds for $f \in L^2_{d\alpha}$ at those x for which $[n\lambda_n(d\mu, x)]^{-1} \leqslant c$. It may be possible to extend this result to multivariate orthogonal polynomials satisfying stronger conditions, for example, functions having certain higher order derivatives and the highest derivative belongs to $L^2_{d\mu}$. However, it seems to be necessary to require the stronger conditions on f through higher order divided differences, which indicates that it may be necessary to consider certain linear combinations of the $(C, 1)$ sums instead of $(C, 1)$ sums themselves. As an alternative and an application of Theorem 5.4, we consider the de la Vallée-Poussin sum

$$V_n(d\mu, f) = n^{-1} \sum_{k=n}^{2n-1} S_k(d\mu, f),$$

which is easier to handle since it preserves polynomials in Π^d_{n-1}.

THEOREM 5.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure, and $\Omega := \operatorname{supp} \mu$ be a compact subset $\subseteq [-1, 1]^d$. Let $\mathbf{x} = (x_1, ..., x_d) \in [-1, 1]^d$ be fixed and let (5.4) and (5.2) hold at $\mathbf{x}$. Then in the interior of Ω*

$$\frac{1}{n} \sum_{k=n}^{2n-1} |S_k(d\mu, f, \mathbf{x}) - f(\mathbf{x})| \leqslant c n^{(d-1)/2} E_{n-1}(f)_\infty. \tag{5.8}$$

In particular, if for all odd d, $f \in C^{(d-1)/2}(\Omega)$, and for all even d, $f \in C^{(d-2)/2}(\Omega)$ and each of its $[(d-2)/2]$ partial derivatives satisfy

$$|D^{\mathbf{k}} f(\mathbf{x}) - D^{\mathbf{k}} f(\mathbf{y})| \leqslant c h^\beta, \qquad |\mathbf{x}-\mathbf{y}|_2 \leqslant h, \qquad |\mathbf{k}| = \frac{d-2}{2}$$

for some $\beta > 1/2$, then in the interior of Ω

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=n}^{2n-1} |S_k(d\mu, f, \mathbf{x}) - f(\mathbf{x})| = 0. \tag{5.9}$$

Moreover, if Δ is a subset of Ω for which $\min_{\mathbf{x}\in\Delta}\{1-x_i^2\}\geqslant\delta>0$, where i is the index in (5.4), and (5.2) holds uniformly on Δ, then both (5.8) and (5.9) hold uniformly on Δ.

Proof. By Theorem 5.4 and condition (5.2) we have

$$\frac{1}{n}\sum_{k=n}^{2n-1}|S_k(d\mu,f,\mathbf{x})| = 2\frac{1}{2n}\sum_{k=0}^{2n-1}|S_k(d\mu,f,\mathbf{x})| - \frac{1}{n}\sum_{k=0}^{n-1}|S_k(d\mu,f,\mathbf{x})|$$
$$\leqslant c[2n\Lambda_{2n}(d\mu,\mathbf{x})]^{-1/2}\|f\|_{\infty,\Omega}\leqslant cn^{(d-1)/2}\|f\|_{\infty,\Omega}.$$

Let $P_n\in\Pi_{n-1}^d$ such that $E_{n-1}(f)_{\infty,\Omega}=\|f-P_n\|_{\infty,\Omega}$. Since $S_n(d\mu,f)$ preserves polynomials in Π_{n-1}^d, we obtain by using the previous inequality with f replaced by $f-P_n$ that

$$\frac{1}{n}\sum_{k=n}^{2n-1}|S_k(d\mu,f,\mathbf{x})-f(\mathbf{x})|\leqslant\frac{1}{n}\sum_{k=n}^{2n-1}|S_k(d\mu,f-P_n,\mathbf{x})|+|f-P_n|$$
$$\leqslant cn^{(d-1)/2}E_{n-1}(f)_{\infty},$$

If d is odd and $f\in C^{(d-1)/2}$, then by [8, p. 90] it is easy to see that

$$E_n(f)_\infty\leqslant cn^{-(d-1)/2}\sum_{|\mathbf{k}|=(d-1)/2}\omega\left(D^{\mathbf{k}}f,\frac{1}{n}\right)_\infty.$$

where $\omega(f,h)_\infty$ is the modulus of continuity of f

$$\omega(f,h)_\infty=\max_{1\leqslant i\leqslant d}\ \max_{|x_i-y_i|\leqslant h}|f(\mathbf{x})-f(\mathbf{y})|.$$

Since $\omega(D^{\mathbf{k}}f,n^{-1})_\infty\to 0$ for $|\mathbf{k}|=(d-1)/2$ as $n\to\infty$, we have (5.8) for odd d. Similarly, for even d we have

$$E_n(f)_\infty\leqslant cn^{-(d-2)/2}\sum_{|\mathbf{k}|=(d-2)/2}\omega\left(D^{\mathbf{k}}f,\frac{1}{n}\right)_\infty.$$

Since our assumptions imply that for even d,

$$\omega\left(D^{\mathbf{k}}f,\frac{1}{n}\right)_\infty\leqslant cn^{-\beta},\qquad|\mathbf{k}|=\frac{d-2}{2},\qquad\beta>1/2,$$

we have (5.9) for even d as well. ∎

Using the estimate of the Christoffel function in Section 3, several consequences of Corollary 5.3 and Theorem 5.5 can be formulated for measures satisfying condition (5.2). Interesting examples include the classical measures mentioned in Section 3 and measures that have support on

$[-1, 1]^d$ and satisfy condition (4.3.1). We shall not formulate these corollaries formally. We note that for $d=1$, the above theorem is the classical one (cf. [2, p. 157])

$$\frac{1}{n}\sum_{k=n}^{2n-1} |S_k(d\mu, f, \mathbf{x})| \leq cE_{n-1}(f)_\infty .$$

For $d=2$, this theorem implies that (5.9) holds for all $f \in C(\Omega)$ satisfying the Lipschitz condition

$$|f(\mathbf{x})-f(\mathbf{y})| \leq h^\beta, \qquad |\mathbf{x}-\mathbf{y}|_2 \leq \frac{1}{n}, \qquad \beta > 1/2.$$

The convergence in this theorem is the so-called strong convergence, since it is clearly stronger than the usual convergence $|V_n(d\mu, f, \mathbf{x}) - f(\mathbf{x})| \to 0$. It is possible that the conditions for the usual convergence will be considerably weaker. We end this paper with the following remark. Because of Corollary 5.3, Theorem 5.5, and the theory for $d=1$, it seems to be reasonable to conjecture that for $f \in L^2_{d\mu}$ the (C, β) mean of orthogonal polynomials in d variables will be strongly convergent under the assumption on μ similar to Corollary 5.3 and Theorem 5.5 for β large enough. The value of β should depend on the dimension d. For example, for $d=2$, $\beta > 3/2$ seems likely. Similar phenomena appear in the convergence of the spherical means of multivariate Fourier series (cf. [12]). However, for the higher order Cesáro means the method used in this paper does not seem to be strong enough. This method depends on the condition (5.2), which is the (C, 1) boundedness of orthogonal polynomials.

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References

1. A. Erdélyi *et al.*, "Higher Transcendental Functions," Vol. 2, McGraw-Hill, New York, 1953.
2. G. Freud, "Orthogonal Polynomials," Pergamon, New York, 1971.
3. J. G. Herriot, Nörlund summability of multiple Fourier series, *Duke Math. J.* **11** (1944), 735–754.
4. A. N. Kolmogorov and S. V. Formin, "Introductory Real Analysis," Dover, New York, 1975.
5. M. A. Kowalski, The recursion formulas for orthogonal polynomials in *n* variables, *SIAM J. Math. Anal.* **13** (1982), 309–315.

6. M. A. Kowalski, Orthogonality and recursion formulas for polynomials in n variables, *SIAM J. Math. Anal.* **13** (1982), 316–323.
7. H. L. Krall and I. M. Sheffer, Orthogonal polynomials in two variables, *Ann. Mat. Pura Appl. (4)* **76** (1967), 325–376.
8. G. G. Lorentz, "Approximation of Functions", 2nd ed., Chelsea, New York, 1986.
9. A. Máté, P. Nevai, and V. Totik, Szegő's extremum problem on the unit circle, *Ann. of Math.* **134** (1991), 433–453.
10. I. P. Mysovskikh, "Interpolatory Cubature Formulas," Nauka, Moscow, 1981.
11. P. Nevai, Géza Freud, orthogonal polynomials, and Christoffel functions: A case study, *J. Approx. Theory* **48** (1986), 3–167.
12. E. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton Univ. Press, Princeton, NJ, 1971.
13. P. K. Suetin, "Orthogonal Polynomials in Two Variables," Nauka, Moscow, 1988.
14. Y. Xu, On multivariate orthogonal polynomials, *SIAM J. Math. Anal.* **24** (1993), 783–794.
15. Y. Xu, Multivariate orthogonal polynomials and operator theory, *Trans. Amer. Math. Soc.* **343** (1994), 193–202.
16. Y. Xu, Gaussian cubature and bivariate polynomial interpolation, *Math. Comp.* **59** (1992), 547–555.
17. Y. Xu, Unbounded commuting operators and multivariate orthogonal polynomials, *Proc. Amer. Math. Soc.* **119** (1993), 1223–1231.
18. Y. Xu, Block Jacobi matrices and zeros of multivariate orthogonal polynomials, *Trans. Amer. Math. Soc.* **342** (1994), 855–866.
19. Y. Xu, Recurrence formulas for multivariate orthogonal polynomials, *Math. Comp.* **62** (1994), 687–702.
20. A. Zygmund, "Trigonometric Series," Vol. 1, Cambridge Univ. Press, Cambridge, 1950.

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